

# STABLE, ROBUST TRACKING BY SLIDING MODE CONTROL \*

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*Abstract:* Sliding mode control is examined from the perspective of obtaining stable and robust tracking of an arbitrary time-varying reference by a multi-input-output, linear, time-invariant system driven by a certain class of bounded errors, nonlinearities and disturbances. Most existing schemes for such systems are subsumed by the one presented here. The results are developed via the use of inverse models, and make clear the constraints imposed by the finite and infinite zero structure of the system. In particular, stable and robust tracking is shown to be obtained by the scheme in this paper if and only if the system is minimum phase.

*Keywords:* Sliding modes, Variable structure systems, Tracking, Robust control, Inverse models, Zeros.

## 1. Introduction

Variable structure systems with sliding modes have been the focus of a growing literature, see for example [1]-[6] and references therein. The results in this paper are developed in the context of stable and robust tracking of an arbitrary time-varying reference by a multivariable, linear, time-invariant system driven by a certain class of errors, nonlinearities and disturbances. Most existing sliding mode schemes for such systems are subsumed by the one described here. Our treatment is built around the use of inverse models, and offers some new insights into the constraints imposed by the finite and infinite zero structure of the system. It is shown that stable and robust tracking is obtained by this sliding scheme if and only if the system is minimum phase. The blending of state-space and polynomial matrix descriptions allows our results to be derived cleanly and generally.

**Stable, Robust Tracking.** Our starting point is the system

$$\dot{x}(t) = Ax(t) + Bu(t) + v(x, t) \quad (1a)$$

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$$y(t) = Cx(t) \quad (1b)$$

where  $x$  is the  $n$ -dimensional state vector and  $u, y$  are  $m$ -dimensional control input and tracking output vectors respectively. The vector  $v$  represents model errors, nonlinearities, and disturbances. (The time argument  $t$  will be dropped for notational simplicity wherever it is not needed for clarity or emphasis.) One can obviously not ask for more independently tracking outputs than there are control inputs. On the other hand, if the number of desired tracking outputs is fewer than the number of control inputs, then there are additional degrees of freedom that may be exploited, and some remarks on the possibilities are made in the Conclusion of this paper.

The control objective is to pick  $u$  as a function of the measured  $x$  so as to have

$$e(t) = y(t) - y_d(t) \quad (2)$$

go to zero, where  $y_d$  is the desired output, i.e., the tracked reference, and is assumed to be sufficiently differentiable — the precise degree of differentiability will become explicit later. We also require that the dynamics of  $x$  be independent of  $v$  — which gives rise to the label “robust” — and that when  $y_d = 0$  identically then  $x$  goes asymptotically to zero — hence the label “stable”.

**Assumptions.** Assume that the system (1) is controllable by  $u$  and observable from  $y$ . Also impose the standard *matching* restriction, [2],[4], on  $v$ , namely

$$v(x, t) = Bw(x, t) \quad (3)$$

for some  $w$  that has a known bound in some norm. It is these assumptions on  $v$  that end up enabling robust tracking. Finally, assume that the transfer function matrix

$$G(\lambda) = C(\lambda I - A)^{-1}B \quad (4)$$

of the system (1) is invertible (as a rational matrix). This assumption is clearly necessary if tracking of arbitrary references is desired.

The invertibility assumption on the transfer function implies that it has as many zeros as poles, if zeros at infinity are included and if multiplicities are properly accounted for, [7]. The controllability and observability of the model imply that this number (of poles and of zeros) is  $n$ , the order of the system (1). The finite zeros are (see [8, Sec. 6.5]) precisely the roots of  $\det(\lambda I - A) \cdot \det G(\lambda)$ . If  $\ell$  denotes the relative order of  $\det G(\lambda)$ , i.e. its denominator degree minus its numerator degree, it follows that there must be  $n - \ell$  finite zeros. Hence the total number of infinite zeros is  $\ell$ .

**Sketch of Main Results.** First consider the *special case where  $CB$  is invertible*. Suppose the control  $u$  is chosen to bring the system in finite time from any initial state to the  $(n - m)$ -dimensional surface given by  $s(x, t) = Cx - y_d = y - y_d = e = 0$  and to then keep the trajectory confined to — or “sliding” on — this surface by switching appropriately, so that  $e = 0$  from then on. That this can always be done under the above assumptions is well established in the literature on sliding modes, see [1]-[6], so tracking is indeed achieved in this case.

To determine the input and state trajectories when the system is sliding, one can use the method of *equivalent control* [1],[3]. This effectively sets  $ds/dt = 0$  for motion on the sliding surface and solves for  $u_{eq}$ , which is the Filippov average of the switched control, [6]:

$$ds/dt = \dot{y} - \dot{y}_d = C\dot{x} - \dot{y}_d = CAx + CB(u_{eq} + w) - \dot{y}_d = 0 \quad (5)$$

from which, since  $CB$  is assumed invertible,

$$u_{eq} = -w - (CB)^{-1}(CAx - \dot{y}_d) \quad (6)$$

The system behavior in the sliding regime is therefore given by

$$\begin{aligned} \dot{x} &= Ax + B(u_{eq} + w) \\ &= (I - B(CB)^{-1}C)Ax + B(CB)^{-1}\dot{y}_d, \end{aligned} \quad (7)$$

and is unaffected by  $w$ . Note that we have *not* guaranteed robustness to  $w$  during the interval when the system trajectory is moving towards the sliding surface but not yet sliding. This problem can be overcome by ensuring that the system *starts* in the sliding mode, and this can be achieved by choosing  $y_d(0) = Cx(0)$ .

Though we have managed to achieve robust tracking in the sliding mode, the dynamics in (7) is evidently fixed, and may be unsatisfactory. The characteristic polynomial of the system in (7) is easily seen to be

$$\det[\lambda I - A + B(CB)^{-1}CA] = \det(CB)^{-1} \cdot \lambda^m \cdot \det(\lambda I - A) \cdot \det G(\lambda) \quad (8)$$

Note from this that the matrix in (7) has  $m$  eigenvalues at 0, corresponding to the fact that the trajectory is confined to an  $(n - m)$ -dimensional surface. The remaining eigenvalues are seen from (8) to occur at the  $n - m$  finite zeros of the transfer function  $G(\lambda)$ . This may have been anticipated by considering the case where  $y_d = 0$  for all time: evidently the equivalent control must in this case be exciting the system at the frequencies of its zeros in order to keep  $y = 0$  when  $u_{eq} \neq 0$ . (This observation has also been made in [9].) Hence stable tracking is obtained with this scheme if and only if the system is minimum phase.

When  $CB$  is *not invertible*,  $u_{eq}$  cannot be determined by simply taking the first derivative of  $y$  as in (5). A more elaborate procedure, familiar from the area of *system inversion*, [10]-[12], is now required. The procedure in effect takes appropriate linear combinations of further derivatives of the components of  $y$  to compute  $u_{eq}$ . However, the equivalent control method requires that all of this procedure be folded into the simple act of computing  $ds/dt$ . This implies that the sliding surface above has to be modified, and we shall show that it goes from  $s = e = Cx - y_d = 0$  to  $s = P(d/dt)e = Lx - P(d/dt)y_d = 0$  for some appropriately chosen *polynomial matrix*  $P(\lambda)$  and corresponding coefficient matrix  $L$ .

It will turn out under our assumptions that  $LB$  is invertible, so the earlier analysis can be followed, with  $C$  replaced by  $L$  and with other obvious changes. Now the tracking error  $e$  in the sliding regime has dynamics that is governed by the equation

$$P(d/dt)e = 0 \quad (9)$$

The associated characteristic polynomial  $\det P(\lambda)$  will be shown to have degree  $\ell - m$  and to be *arbitrarily assignable*. We shall also demonstrate that the dynamics of  $x$  is unaffected by  $w$  in the sliding mode. The eigenvalues governing the state dynamics will be seen to comprise precisely the  $\ell - m$  roots of  $\det P(\lambda)$ , the  $n - \ell$  finite zeros of  $G(\lambda)$ , and  $m$  eigenvalues at 0 that correspond as before to the confinement of the trajectories to the  $(n - m)$ -dimensional sliding surface. Again, the appearance of the finite zeros of the system may have been anticipated from the fact that, with  $y_d = 0$  identically, the output  $y$  can be made to go to 0 arbitrarily fast in this scheme.

**Outline of Paper.** The remainder of the paper fills out and proves the above results. Section 2 contains the main results. Inverse models are introduced in a way that is convenient for our purposes, and their role in defining sliding surfaces and the associated sliding dynamics is established. This section also briefly considers how to pick control laws that cause trajectories to reach a sliding surface in finite time. Section 3 presents some illustrative examples. The Conclusion of the paper mentions possible extensions. Certain proofs whose details are not central to the flow of the paper are gathered in Appendices A1-A3.

## 2. Sliding Modes Defined Through Inverse Models

**Inverse Models.** The literature on system inversion, see [10]-[12] for instance, shows how one may compute (via the structure algorithm in [10] for example) an invertible *polynomial matrix*  $Q(\lambda)$  such that

$$Q(\lambda)G(\lambda)\Big|_{\lambda=\infty} = I \quad (10)$$

In the language of [13],  $Q(\lambda)$  is an "identity interactor" for the system (1) or the transfer function (4). The reason that  $Q(\lambda)$  appears explicitly or implicitly in all approaches to system

inversion is that the operator  $Q(d/dt)$ , acting on the system output, takes linear combinations of derivatives of the components of this output vector to produce a signal that directly contains the input. This is suggested by (10) but will be verified more precisely by (14) below.

It can be seen from (10) that the relative order of  $\det Q(\lambda) \cdot \det G(\lambda)$  is 0. Since the relative order of  $G(\lambda)$  is  $\ell$ , it follows that the degree of the polynomial  $\det Q(\lambda)$  is also  $\ell$ . It is further true that the detailed polar structure of  $Q(\lambda)$  at  $\lambda = \infty$ , in the sense of [7], is identical to the zero structure of  $G(\lambda)$  at  $\lambda = \infty$  or the polar structure of  $G^{-1}(\lambda)$  at  $\lambda = \infty$ . In particular, the highest power of  $\lambda$  in  $Q(\lambda)$  equals its highest power in  $G^{-1}(\lambda)$ .

The  $Q(\lambda)$  in (10) is not unique, however. It is not hard to show that if  $\bar{Q}(\lambda)$  is one particular solution of (10), then a polynomial matrix  $Q(\lambda)$  is a solution if and only if

$$Q(\lambda) = \bar{Q}(\lambda) + N(\lambda) \quad (11)$$

where  $N(\lambda)$  is a polynomial matrix such that  $N(\lambda)\bar{Q}^{-1}(\lambda)$  is strictly proper (i.e. vanishes at  $\lambda = \infty$ ). A necessary condition on  $N(\lambda)$  for this strict properness is that its column degrees (i.e. the highest degree among the entries in each of its columns) be less than the corresponding column degrees of  $\bar{Q}(\lambda)$ . It follows from (11) that all solutions  $Q(\lambda)$  have the same column degrees. This simple condition for strict properness is also sufficient if and only if  $\bar{Q}(\lambda)$  is "column reduced", i.e. if and only if the degree  $\ell$  of  $\det \bar{Q}(\lambda)$  equals the sum of the column degrees of  $\bar{Q}(\lambda)$ , see [8, Sec. 6.3].

We now examine more precisely the role of  $Q(\lambda)$  in defining an inverse model for the given system (1), (3). It is shown in Appendix A1 that

$$I - Q(\lambda)G(\lambda) = H(\lambda I - A)^{-1}B \quad (12)$$

for some constant matrix  $H$ . The controllability of the system guarantees that  $(\lambda I - A)^{-1}B$  has no constant vectors in its left nullspace, so  $H$  can be found as the unique constant solution of (12) for a given  $Q(\lambda)$ . (Another route to computation of  $H$  is provided by the polynomial matrix division described in Appendix A1, while a third approach is contained in (23b) below.) On application of the well-known matrix inversion lemma to (12), one obtains the following expression for the transfer function of the inverse system

$$G^{-1}(\lambda) = [H(\lambda I - A - BH)^{-1}B + I]Q(\lambda) \quad (13)$$

Note that the zeros of  $G(\lambda)$  at finite values of  $\lambda$  give rise to poles of  $G^{-1}(\lambda)$  at the same locations. Since  $Q(\lambda)$  has no finite poles, (13) shows that these poles must be eigenvalues of  $A + BH$ . In other words, *the eigenvalues of  $A + BH$  must include all the finite zeros of  $G(\lambda)$ .*

Given the expression in (13) for the inverse of the system transfer function, it is perhaps not surprising that the input  $u + w$  of the system (1), (3) can in fact be determined from the output  $y$  via the *inverse model*

$$\dot{x} = (A + BH)x + BQ(d/dt)y \quad (14a)$$

$$u + w = Hx + Q(d/dt)y \quad (14b)$$

where the state vector  $x$  is the same as that in (1). A proof of this is given in Appendix A3. The inverse model (14) takes as its input the signal  $Q(d/dt)y$ , which constitutes linear combinations of derivatives of components of the output of the original system (1). The output of the inverse model (14) is the input  $u + w$  of the original system (1), (3). This confirms the claim following (10) regarding the significance of  $Q(\lambda)$ .

**The Sliding Surface.** The inverse model (14) shows that  $u + w$  can be computed from  $Q(d/dt)y$ . Recall from the sketch in the Introduction that the equivalent control method computes  $u_{eq}$  by setting  $ds/dt = 0$ . This suggests using a sliding surface with the property that  $ds/dt = Q(d/dt)e$ , which can be achieved by picking

$$s = P(d/dt)e = 0 \quad (15)$$

where  $P(\lambda)$  is a *polynomial* matrix that satisfies

$$\lambda P(\lambda) = Q(\lambda) \quad (16)$$

It is obvious from (16) that such a  $P(\lambda)$  exists if and only if

$$Q(0) = 0 \quad (17)$$

Since the degree of  $\det Q(\lambda)$  is  $\ell$ , that of  $\det P(\lambda)$  is seen from (16) to be  $\ell - m$ . The number of degrees of freedom available in picking  $P(\lambda)$  and hence the sliding surface will be shown in the next subsection to be  $m(\ell - m)$ .

With  $s$  chosen as in (15), (16),

$$ds/dt = (d/dt)P(d/dt)e = Q(d/dt)e = Q(d/dt)y - Q(d/dt)y_d \quad (18)$$

as desired. Now setting  $ds/dt = 0$  in the sliding mode, as prescribed by the equivalent control method, and using the inverse model (14b) to substitute for  $Q(d/dt)y$ , we find that

$$u_{eq} = -w + Hx + Q(d/dt)y_d \quad (19)$$

With this control input in (1), (3), the state dynamics in the sliding mode is governed by

$$\dot{x} = (A + BH)x + BQ(d/dt)y_d \quad (20)$$

and the output tracking error satisfies (15), (9). Once again, therefore, we obtain robust tracking after sliding is established.

Before proceeding further, we need to show that (15), (16) does actually define a surface in state space. Appendix A3 demonstrates that

$$P(d/dt)y = P(d/dt)Cx = Lx \quad (21)$$

for a constant matrix  $L$  that is uniquely determined by  $P(\lambda)$ , so

$$s(x, t) = P(d/dt)y - P(d/dt)y_d = Lx - P(d/dt)y_d \quad (22)$$

Hence (15) does indeed constitute a surface in state space. To compute  $L$  from (21), one evaluates  $P(d/dt)Cx$ , with repeated elimination of  $\dot{x}$  using (1a) — this is what the polynomial matrix division in Appendix A1, used in Appendix A3, effectively does. The derivation in Appendix A3 guarantees that  $u + w$  and its derivatives will not appear in the result, so the final result is of the form  $Lx$ .

The matrices  $H$  and  $L$  are seen from Appendices A1 and A3 to be related by

$$LB = I \quad (23a)$$

$$H = -LA \quad (23b)$$

so  $L$  is a left inverse of  $B$ . It should *not* be inferred that *every* choice of left inverse  $L$  leads to a sliding surface. Indeed, we have already claimed above (and will show below) that there are only  $m(\ell - m)$  degrees of freedom in picking the sliding surface, while there are  $m(n - m)$  degrees of freedom in picking left inverses of  $B$ . Once  $L$  is found from (21) or an equivalent computation, then  $H$  may be found from (23b), as mentioned earlier.

With  $s$  identified as in (22), the development in (5)-(7) can be followed, with  $C$  and  $y_d$  replaced by  $L$  and  $P(d/dt)y_d$  respectively. Using the relations in (23), the result is again (20), as expected, which is the generalization of (7). Note that in order to ensure that the system starts directly in the sliding mode, one must pick  $y_d$  and its derivatives such that  $s(x(0), 0) = 0$ .

**Dynamics in the Sliding Mode.** Without loss of generality, the tracked reference  $y_d$  can and will be taken to be 0 when dealing in this subsection with stability and dynamics. The state dynamics (20) in the sliding regime is then undriven. Note from (15) that the dynamics of the tracking error in the sliding mode is governed by the matrix  $P(\lambda)$  and the associated characteristic polynomial  $\det P(\lambda)$ , whose degree we have already identified as  $\ell - m$ . Taking into account the parametrization of  $Q(\lambda)$  in (11) and the relation between  $P(\lambda)$  and  $Q(\lambda)$  in

(16), it is seen that a complete parametrization of  $P(\lambda)$  in terms of a particular solution  $\bar{P}(\lambda)$  is given by

$$P(\lambda) = \bar{P}(\lambda) + M(\lambda) \quad (24)$$

where  $M(\lambda)$  is a polynomial matrix such that  $M(\lambda)P^{-1}(\lambda)$  is strictly proper. Since the column degrees of  $M(\lambda)$  are less than the corresponding degrees of  $\bar{P}(\lambda)$ , it follows that the column degrees of  $P(\lambda)$  are invariant. Hence the orders of the highest derivatives of the components of  $e$  required to form the sliding surface are invariant.

The question now is what freedom there is in assigning the tracking error dynamics. Combining results in [8, Sec. 6.3, Sec. 7.2] with the parametrization in (24) shows that (24) affords us  $m(\ell - m)$  degrees of freedom, and that we can in particular use this freedom to make (in many ways, if  $m > 1$ ) the polynomial  $\det P(\lambda)$  equal to *any* polynomial of degree  $\ell - m$ , up to a nonzero constant multiple. Arbitrarily fast decay of the output tracking error may therefore be obtained (assuming, as in our problem formulation, that the control input is not constrained).

We turn from examination of the output tracking error dynamics to that of the system state in the sliding regime. Since  $y$  cannot display dynamic behavior that is not present in  $x$ , it must be true that the eigenvalues of  $A + BH$  in (20) include all the  $\ell - m$  roots of  $\det P(\lambda)$ . Also, we have already argued that the matrix  $A + BH$  must contain all the  $n - \ell$  finite zeros of the given system (1) among its eigenvalues. Finally,  $A + BH$  must have  $m$  eigenvalues at the origin, corresponding to the confinement of the trajectories to the  $(n - m)$ -dimensional subspace  $Lx = 0$ . All of this therefore accounts for  $[\ell - m] + [n - \ell] + m = n$  eigenvalues of  $A + BH$ , and there are no more to account for! This can be explicitly and directly shown by verifying that

$$\det(\lambda I - A - BH) = \det P(\lambda) \cdot \lambda^m \cdot \det(\lambda I - A) \cdot \det G(\lambda) \quad (25)$$

using the various relationships already established. In conclusion:

*The dynamic behavior of the system on the sliding surface is governed by  $n - \ell$  eigenvalues that are located at the finite zeros of the system and a further  $\ell - m$  eigenvalues that are freely assignable, subject to being a self-conjugate set.*

It follows that stable tracking is possible with this scheme if and only if the system is minimum phase.

**Obtaining Sliding.** The development so far has focused on what happens once the sliding mode is attained. In order to see what sort of nonlinear control law is needed to reach and stay on the sliding surface, note from (14b) and (18) that

$$ds/dt = Q(d/dt)y - Q(d/dt)y_d = u + w - Hx - Q(d/dt)y_d \quad (26)$$



Now pick the control law to be

$$u = f(s) + Hx + Q(d/dt)y_d \quad (27)$$

so that

$$ds/dt = f(s) + w \quad (28)$$

The function  $f(s)$  can be picked so as to guarantee convergence of  $s$  in (28) to 0 in finite time in the presence of a bounded  $w$ . Many ways for doing this, using switching control laws, are discussed in the sliding mode literature, and so will not be considered here. It should be noted that our formulation has not considered the bounds on control that will exist in any realistic situation, and these bounds must be explicitly reckoned with in any actual design. The effect of the choice of  $Q(\lambda)$  (and hence  $H$ ), of  $y_d$ , and of  $f(s)$  on the control  $u$  must be assessed from (27).

One final point is worth making. The sliding surface specified by  $s = 0$  can also be specified by the condition  $Ts = 0$  where  $T$  is any nonsingular  $m \times m$  matrix. Working with the sliding surface specified in this form does not change any of the expressions for motion *while* sliding, though it will change the expressions that govern the *approach* to the sliding surface.

### 3. Examples

**Example 1.** If  $CB$  is invertible, the system (1) has  $\ell = m$  infinite zeros and  $n - m$  finite ones. The set of  $Q(\lambda)$  satisfying (10), (11) is given precisely by  $Q(\lambda) = (CB)^{-1}(\lambda I + N)$ , where  $N$  is an arbitrary constant matrix. Imposing the additional constraint in (17) that  $Q(0) = 0$  yields  $Q(\lambda) = (CB)^{-1}\lambda$  as the only solution. The corresponding  $P(\lambda)$  is simply  $(CB)^{-1}$ . It follows that the sliding surface can be defined via  $(CB)^{-1}(Cx - y_d) = 0$  or equivalently via  $Cx - y_d = 0$ , which is the same surface as the one considered in the Introduction. The results of the Introduction are recovered on applying (15)-(20). ••

**Example 2.** Suppose the model (1) has order  $n = 2m$ , and that its coefficient matrices are given by

$$A = \begin{pmatrix} 0 & I_m \\ -J^{-1}K & -J^{-1}D \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ J^{-1} \end{pmatrix}, \quad C = (I_m \quad 0) \quad (29)$$

This model represents the prototype spring-inertia system, with  $J$ ,  $K$  and  $D$  being inertia, spring and damping coefficient matrices respectively. Such models are found in descriptions of robot manipulator dynamics [5],[6], in the swing equations that describe electromechanical oscillations of power systems, and so on. The first block of state variables comprises position variables  $\delta$ , which constitute the outputs in this example, the second block comprises velocity variables  $\omega$ , and the inputs are generalized forces.

For this model,

$$G(\lambda) = (\lambda^2 J + D\lambda + K)^{-1} \quad (27)$$

All  $n = 2m$  zeros of this system are at  $\infty$ , so  $\ell = 2m$ . The set of  $Q(\lambda)$  in (10), (11) is precisely given by  $Q(\lambda) = J(\lambda^2 I + M\lambda + N)$ , where  $M$  and  $N$  are arbitrary constant matrices. Imposing the constraint (17) yields  $N = 0$ . We then have  $P(\lambda) = J(\lambda I + M)$ . The roots of  $\det P(\lambda)$  are evidently the  $m$  eigenvalues of  $M$ , and can be assigned to any self-conjugate set of values. The corresponding sliding surface is given by  $(\omega - \dot{\delta}_d) + M(\delta - \delta_d) = 0$ . ••

**Example 3.** Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad (28a)$$

$$C = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \quad (28b)$$

so

$$G(\lambda) = \begin{pmatrix} (\lambda+2)/(\lambda+1)^2 & 1/\lambda \\ (\lambda+2)/(\lambda+1)^2 & 1/(\lambda+1) \end{pmatrix} \quad (29)$$

Using the methods described in [7], this transfer function matrix can be seen to have: 1 first-order pole at  $\lambda = 0$ ; 1 first-order and 1 second-order pole at  $\lambda = -1$ ; 1 first-order zero at  $\lambda = -2$ ; and 1 first-order and 1 second-order zero at  $\lambda = \infty$ . The total number of infinite zeros  $\ell$  is thus 3. One can show that the general form for  $Q(\lambda)$  is

$$Q(\lambda) = \begin{pmatrix} -\lambda^2 + n_1\lambda + n_2 & \lambda^2 + (1 - n_1)\lambda + n_5 \\ \lambda^2 + n_3\lambda + n_4 & -\lambda^2 - n_3\lambda + n_6 \end{pmatrix} \quad (30)$$

for arbitrary  $n_i$ . Imposing the condition in (17) and computing  $P(\lambda)$  according to (16) yields

$$P(\lambda) = \begin{pmatrix} -\lambda + n_1 & \lambda + (1 - n_1) \\ \lambda + n_3 & -\lambda - n_3 \end{pmatrix} \quad (31)$$

The determinant of this matrix is  $-(\lambda + n_3)$ , which is of degree 1 and arbitrarily assignable, as expected. The corresponding sliding surface, taking  $n_1 = 0$  for instance, can then be computed from (21), (22) to be

$$\begin{pmatrix} 2x_1 + x_2 - \dot{y}_{1d} + \dot{y}_{2d} + y_{2d} \\ (1 - n_3)x_3 + n_3x_4 + \dot{y}_{1d} + n_3y_{1d} - \dot{y}_{2d} - n_3y_{2d} \end{pmatrix} = 0 \quad (32)$$

••

## Conclusion

Most existing approaches to sliding mode control have *not* considered the tracking problem posed here. The present perspective is more useful in many situations, and includes the main results developed via other approaches. If no tracking outputs are specified *a priori*, then one is free to choose  $C$  to obtain desirable system dynamics. This is the form in which the sliding mode control problem is more commonly considered in the literature. The results in this paper show that  $C$  should be picked to place any finite transmission zeros at desirable locations. Rosenbrock's theorem [14, Ch.5, Thm.4.1] indicates the freedom available in doing this. If some, but fewer than  $m$ , tracking outputs are designated, one has the nontrivial problem of augmenting the output set to obtain a  $C$  that leads to zeros at desirable locations. Some discussion of such problems is contained in [15]. Note that any zeros of the original system before augmentation will remain as zeros of the augmented system.

One can examine extensions of the results here to the nonlinear case, building on the notions of left invertibility for such systems, [16]. Exploration in this direction may be found in [17].

**Appendix A1.** Note from (10) that

$$Q(\lambda)C(\lambda I - A)^{-1}B = I + R(\lambda) \quad (\text{A1.1})$$

for some strictly proper  $R(\lambda)$ . Carrying out division [8, Sec. 6.3] of  $Q(\lambda)C$  by  $\lambda I - A$  yields

$$Q(\lambda)C = L(\lambda)(\lambda I - A) - H \quad (\text{A1.2})$$

where  $L(\lambda)$  is a polynomial matrix and  $H$  is a constant matrix, both uniquely determined. Now (A1.1) becomes

$$[L(\lambda) - H(\lambda I - A)^{-1}]B = I + R(\lambda) \quad (\text{A1.3})$$

Equating strictly proper parts on both sides of (A1.3) and substituting for  $R(\lambda)$  in (A1.1) establishes (12). Equating polynomial parts shows that  $L(\lambda)$  is a polynomial left inverse of  $B$ :

$$L(\lambda)B = I \quad (\text{A1.4})$$

Denoting  $L(0)$  by  $L$ , (23a) follows from (A1.4).

**Appendix A2.** Using (1), (3), (A1.2) and (A1.4), we see that

$$\begin{aligned} Q(d/dt)y &= Q(d/dt)Cx = L(d/dt)[I(d/dt) - A]x - Hx \\ &= L(d/dt)B(u + w) - Hx = (u + w) - Hx \end{aligned}$$

or  $u + w = Hx + Q(d/dt)y$ , which is (14b), and which yields (14a) on substitution in (1).

**Appendix A3.** From (16) and (A1.2), and recalling that  $L = L(0)$ , it follows that

$$P(\lambda)C = \lambda^{-1}[L(\lambda) - L](\lambda I - A) + L - \lambda^{-1}(LA + H) \quad (\text{A3.1})$$

Now (17) and (A1.2) show that

$$LA + H = 0 \quad (\text{A3.2})$$

(which is (23b)) so the last term in (A3.1) drops out. Also note that the term  $\lambda^{-1}[L(\lambda) - L]$  is polynomial. From (A1.4) we see that

$$[L(\lambda) - L]B = 0 \quad (\text{A3.3})$$

Putting together (1) and (A3.1)-(A3.3) yields (21).

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